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Euclid's Geometry: the Case of Contradiction

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This paper surveys Euclid's geometry. After raising philosophical questions about the relation between the diagrams and the words, the question is raised concerning how there can be a diagram appropriate to a *reductio ad absurdum* proof, which by definition operates with a contradiction. This leads us to discover two different kinds of proof of contradiction, one kind in Euclid's *reductios*, and the other kind features in the images of the Impossible Figures movement.

1. Introduction

Euclid's *Elements* is a monumental achievement, and so early too! It presents us with a wealth of definitions, axioms and especially proofs of Propositions; and, as we will see, gives rise to meta-questions of philosophical interest. We will work our way through some of these, before arriving at a place which enables us to raise questions about an area of geometry, Impossible Figures, which Euclid did not know about, but which his work certainly bears on.

2. What are Euclid's words about?

Inspection of the *Elements* reveals two kinds of content: diagrams, and words including names for parts of diagrams, such as points, lines, circles, areas. The words are used to state and prove Propositions, expressed in words, which are ostensibly about the diagrams.

What are the words really "about"? Words can be about anything and that is their strength. Euclid's words are about diagrams and their parts, it might seem. But what are these? Perhaps perfect Platonic shapes or geometrical forms? Surely not: Platonism might deliver perfect shapes but the acausality of Platonic forms, if acausal they be, cannot account for how we interact with them. Physical shapes on a page? No, if only because physical diagrams, drawn by hand or machine, lack the perfection of form that would seem to be possessed by the subject matter of Euclidean Propositions and proofs.

But there is a more plausible answer. Assuming that space is a real thing, which is a reasonable position with many important supporters from Newton to Nerlich, we can identify perfect lines, circles, spheres and the like, as parts of space. For example, we learn from Descartes' methods that perfect circles around the origin are given as collections of points corresponding to equations of the form $x^2+y^2=r^2$, where r is the radius and x and y are the x- and y- coordinates respectively. In this account, lines and other parts of space are not Platonist universals, they are mereological wholes of points. There is no denying, of course, that there are *prima facie* epistemological problems with these items; but on the other hand postulation of them in causally-relevant physical theory is well-entrenched. Euclid himself defined a point as "that which has no part" (Book 1 Def 1: he meant of course no proper part, but this is not seen in a diagram, since "points" in a drawn diagram have proper parts). He also defined a line as "breadthless length" (Def 2) and "the extremities of a line are points" (Def 3): Euclid seems to have been thoroughly realist about geometrical items, while at the same time denying that they are diagrams and their parts.

3. What are the diagrams for?

To reinforce an earlier point, if Propositions and their proofs are about perfectly-shaped parts of space, then they are not about drawn physical diagrams and their parts, because diagrams are not perfectly shaped. But if that is so, then why are diagrams illuminating? What is their use? It is undoubted that diagrams are illuminating, we have only to imagine the *Elements* with the diagrams removed: something would be grievously lacking.

How are diagrams illuminating? One obvious thing to say is that diagrams *approximate* the perfect shapes in space. They improve the understanding by displaying a shape which is in a natural way *like* the perfect shape. Likeness here would seem to be some sort of root-mean-square deviation: crudely, the more the deviation around some mean, the less the likeness. Importantly, *words are not like diagrams*. They employ a different mode of representation. Nelson Goodman (1968) rather nailed this one, I think: words represent shapes by pure convention, whereas diagrams represent shapes using a different mechanism, *exemplification*, by giving you something *like* what is referred to. Goodman went further in claiming that resemblance was itself conventional, but we will not get entangled in that particular thicket, except to say that with geometrical diagrams, the root-mean-square speculation of likeness maybe has rather better prospects than with pictures in general.

So, Propositions and their proofs need diagrams, in the sense that in understanding a diagram, we have a unique and *less conventional* mode of understanding of the associated Proposition, namely exemplification. The Propositions are of course words themselves. Again, proofs also need words to yield understanding, if only because their related Propositions are stated in words. I am inclined to go further, however, and say that without seeing or imagining a diagram there is no *real* understanding of a Proposition or its proof.

We take note of *the generality problem* about diagrams, which has seemed to some to pose a difficulty. How can one diagram per Proposition be enough? Shapes in space come in different sizes, orientations and with different internal features. No one diagram would apply to them all, surely, even if they are associated with the same Proposition. But this is too quick, surely. It is hardly the case that a diagram is *necessary and sufficient* for a proof, so that, for example, you would need different diagram sizes to illustrate proofs of the one Proposition applying to different sized shapes. If there is no appeal in the Proposition and proof to a particular feature of the diagram (e.g. size, orientation) then this feature can be ignored, that is can be generalised over.

4. To get the right words off the diagram

Apart from being necessary for full understanding, is there any role for a diagram in supplying verifications or justifications to steps in a proof? One useful and influential contribution to this matter is due to Kenneth Manders (1995). He distinguishes **exact** features of a diagram from **co-exact** features.

The distinction is in terms of what it is permitted to be "read off" from a diagram, to be used as a premiss in a proof *without further justification*.

Manders' main point is that exact features (e.g. equality of lengths of lines or angles, straightness of lines, right angles, circles) are too exact for us to be justified in loading a perceptual feature from a diagram to the words of a proof. Contrast this with co-exact features (e.g. inequalities, incidences or proper inclusions) which can be appreciated over a range of variation of features). In Manders' words, co-exact features are "unaffected by some range of every continuous variation of a specified diagram" (1995:92). This is a vaguer or looser feature than exactness, but must surely be necessary if hand-drawn diagrams are to illuminate proofs to the point of justification by perception of particular steps.

An example of an exact feature is given in the very first Book 1 Proposition 1. Euclid shows that *on any (finite) straight line there can be constructed an equilateral triangle with the given line as one of its sides*. The equality of the sides is an exact feature, but we cannot read it off a diagram, we have to prove it.

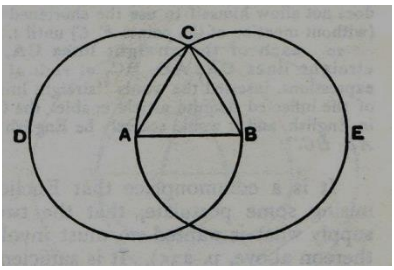

Diagram 1: Book 1, Proposition 1

Proof: Let the given line be AB. Construct a circle with centre A and radius AB. Construct a second circle with centre B and radius AB. The circles intersect in a point C. Join the straight lines AC and BC. The required equilateral triangle is ABC. It is equilateral since AB=AC being both radii of a circle, and BA=BC being radii of the other circle.

Remark: The equality of AB and BA seems to have been taken for granted by Euclid.

An example of a co-exact feature can be found in Book 1 Proposition 16. Euclid shows that, *for any triangle, if one of the sides is extended, the exterior angle is greater than either of the interior and opposite angles.*

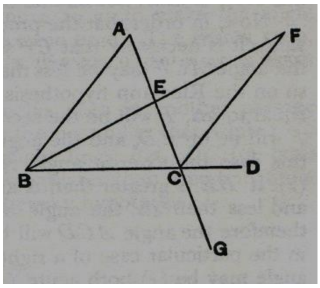

Diagram 2: Book 1, Proposition 16

Proof: Let the given triangle be ABC, and let BC be the extended side, extended to D. Bisect AC at E, and join BE, extending it to BF where BE=EF. Extend AC an arbitrary distance to G. Since AE=EC and BE=EF and angle AEB=angle FEC (Prop 15), the triangles ABE and FEC are congruent (or equal, Euclid: Prop 4). Thus corresponding angles BAE and ECF are equal. But angle ECD > angle ECF (co-exact observation). Hence angle ACD > angle BAE.

Remarks: (1) It is plain that the proof can be re-run for the other sides being extended – thus generality. (2) The co-exact observation about angles at the second-last step also appeals to Euclid's Common Notion 5 "The whole is greater than the part", which however does not mention angles. (3) This step is characteristic of co-exactness: it is the diagram that justifies the step in the argument. (4) Euclid also has a definition of angle, the historical complexities of which we do not pursue (see Heath's careful discussion in Book 1, 176–180).

5. An aside: Venn diagrams

It is useful to have a contrasting perspective on Venn diagrams, which likewise have diagrams plus text, and have a bearing on proofs including Euclidean proofs (for a useful discussion of the differences see Shin & Lemon, 2003). Aristotle's four categorical forms for text were: **All A are B, No A are B, Some A is B**, and **Some A is not B.** Each of these can be represented diagramatically as differing relationships between areas (typically circles) representing the sets (extensions) associated with the terms A and B. Arguments with two premisses and conclusion in the above categorical forms were called *syllogisms*. Venn diagrams facilitate testing the validity and invalidity of arguments in text employing only categorical forms, for any number of premisses from one up including syllogisms. Venn diagrams have topological features, specifically inclusion and exclusion, which enable validity and invalidity to be read off the page.

In this limited sense Venn diagrams can be said to have co-exact features. Hence, one might speculate whether to generalise the concepts of co-exactness and exactness to apply to Venn diagrams as well. Moreover, since Euclidean proofs employ a background logic including *at least* syllogistic logic, it might be argued that a full Euclidean proof *must* include one or more Venn diagrams to justify the logical steps. For a simple example, Heath allows as part of the proof of Proposition 6 below the (valid) one-premiss argument: **All non-A are non-B, so all B are A.**

However, while Venn diagrams can assist following a proof by enabling reading text off the page, the features being described are different from Euclidean proofs, so that generalisation of the concept of co-exactness is not obvious (thanks to a referee for raising this point). Furthermore, while Aristotle himself described the theory of syllogisms, he did not employ Venn diagrams which came many centuries later. That is, it is possible to employ the logical steps in Aristotle's arguments without assistance from Venn diagrams. In contrast, as we have seen, Euclidean diagrams seem to be essential for understanding Euclidean proofs.

6. *Reductio* arguments

Now we are closing in on our main conclusion concerning inconsistency. Manders discusses the role of *reductio* arguments. He sees a *prima facie* problem. How can a diagram illustrate a proof involving the supposition of a *reductio*? After all, a *reductio* begins with an assumption that the aimed-at Conclusion is false, and shows that a contradiction follows. How then is the contra-theorem to be illustrated, if it is never exemplified?

Manders (1995) allows that the diagram for a *reductio* CAN illustrate a contradictory premiss for a *reductio*, but only indirectly. Manders' (1995) idea is that co-exact features must be represented correctly, that is, what is read off from the co-exact features of the diagram is retained in the

Conclusion of the *reductio* argument. In contrast, at least one exact feature can (indeed must!) be incorrectly drawn, that is the diagram lacks the feature attributed by the exact premiss. That way, the task of drawing a contradiction is avoided. One of Manders' (1995) examples is the diagram in Heath of Book 1 Proposition 6. Euclid shows that *if in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.*

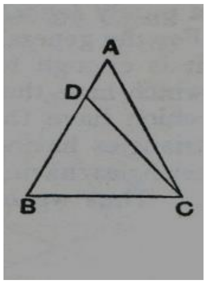

Diagram 3: Book 1, Proposition 6

Proof. Let ABC be a triangle with angle ABC=angle ACB. It is required to prove that side AB=side AC. For suppose not for *reductio.* Then one of the sides is longer than the other (exact feature). Let it be AB > AC. Measure off BD along BA where BD=AC (exact feature, not represented on diagram). Join DC. Now, DB=AC and BC=CB and angle DBC=angle ACB (by stipulation), so then triangle DBC is equal (congruent) to ACB (by Prop 4). But this is absurd since by (co-exact) observation triangle DBC is less than (that is, a proper part of) triangle ACB.

7. Impossible Figures

Manders' (1995) distinctions are useful. But there is more to be said in connection with the important twentieth-century development of **Impossible Figures (IFs).**

The Impossible Figures movement got underway properly in 1934 when the young Swede Oscar Reutersvärd drew what has come to be called somewhat erroneously in the literature, the Penrose Triangle (see the top left of Diagram 5). An important contributor to the movement was the formidable M.C. Escher, but he was not the first. The terminology

"Impossible Figures", IF, is due to Teddy Brunius, and is fairly settled by now, so we stick with that. Even so, the kind of impossibility which is up here, is not mere physical impossibility, that is contrary to natural law, but something stronger: incompatibility at least with logic or mathematics also (it is not intended here to beg the question either for or against logicism).

It will help if look at some examples of IFs. The first example is by Escher's student Bruno Ernst.

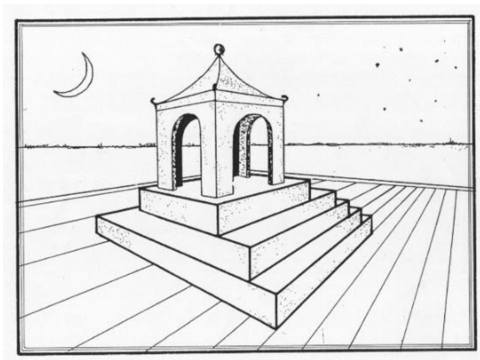

Diagram 4: *The Wearisome and the Easy Ways to the Top*. Bruno Ernst (1984)

Cutting a long story short, it is claimed here that there are five basic forms which can be simplified as follows.

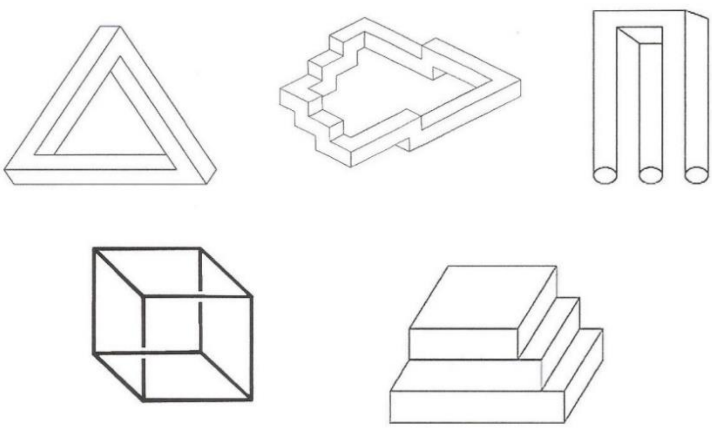

Diagram 5: Five basic forms of Impossible Figures

Of these five forms, schematised from the originals, the top three were first drawn by Oscar Reutersvärd. The top two left, the Penrose Triangle and the Stairway, were re-discovered by Roger Penrose in 1956. The bottom left, the Escher Cube, was drawn by M.C. Escher in his masterpiece *Belvedere* (1958). Bottom right is called the Steps, and is evidently a simplified version of Diagram 4 by Bruno Ernst. It is this which we concentrate on.

Now the claim has been made that these are impossible. But clearly they are sitting there for all to see. So what is impossible about them?

Many pictures have a content, and among contents there are 2-D contents and 3-D contents. 3-D contents are obtained when we project into the third dimension: good examples include perspective and occlusion. The 2-D aspects of the above images are obviously not impossible; but *something* is impossible about them, which indicates that it is the 3-D contents that are impossible. More exactly, what makes it impossible is that it has contradictory 3-D content: the mind projects a contradictory 3-D theory as part of its content, of how it seems. Euclid himself did not shirk the third dimension: Books XI-XII are about solid geometry.

We claim here that these Impossible Figures do not have any existing examples in the 3-D physical world because they have contradictory 3-D content. Here is a quick proof for the case of the Ernst Steps. First the figure is lettered (taken from Mortensen, 2010:130).

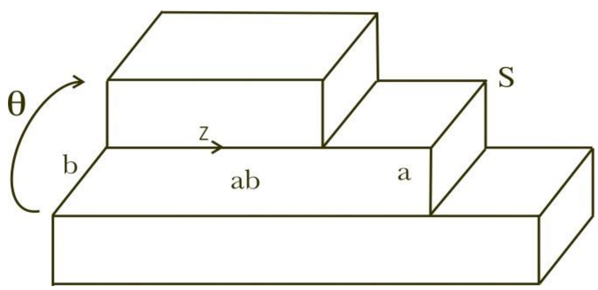

Diagram 6: The Steps lettered

Proof of Contradiction. The argument goes: (1) **a** is vertical, (2) **b** is horizontal, (3) **a** is coplanar with **b**, since both lie in the plane **ab**, and (4) **a** and **b** arbitrarily extended do not meet. Note that the fourth premiss is necessary, since without it (1) – (3) are mutually consistent – a vertical and a horizontal can meet in the one plane, but not if they are skew to each other.

Taking (3) and (4) together, then, we have (5) that **a** and **b** are parallel. But then if **a** is vertical, so must be **b**. That contradicts (2), the horizontality of **b.** Note in passing that these premisses are all part of the 3-D contents of the image, that is why simple 2-D will not do.

This argument can be seen in more detail in Mortensen (2010:130–133), along with arguments for contradictoriness in the cases of the other Forms (Chapters 9–15).

One significant matter in this proof concerns the premiss that **ab** is a plane. That is right about the way the figure *looks*. But if the figure derives from a physical object, then it might be that **ab** is not a plane, but twists from horizontal to vertical without that being noticeable. Then there would be a consistent object as a source of the figure, the way the figure looks would be the way a certain consistent object looks. Indeed, one can see why this can arise in a consistent world: there seems to be a default setting in our perceptual apparatus; if a twist is not perceived, our perceiver defaults to *flat*.

The same can be said for several of the other forms. It is well-known that it is possible to build objects which photograph looking like that (a possible exception is the Fork, top right of Diagram 5). For example, the Triangle can be built not-joined-up; but photographed from a particular angle gives the illusion that it looks joined up. Our perceptual apparatus obviously employs another default mechanism, whereby items at a small angular distance to one another look like they are also at a small radial distance to one another. It is easy to demonstrate this by lining up fingers from opposite hands so that they look touching even though far apart.

But here is an important point: *Reutersvärd was not drawing illusory aspects of existing physical objects: he made his figures up*. In so doing, he showed that it was possible to have contradictory visual content independently of there being any physical objects that might look that way from a preferred aspect. As Reutersvärd noted himself, *his figures were not illusions.*

8. *Reductio* or proof of contradiction?

So we come down to this. What is so special about the above proof of contradiction of The Steps? Is it no more than just another *reductio*? Just a proof that The Steps diagram has no existing exemplar? Is that all there is to Impossible Figures?

Euclid's *reductio* suppositions were not intended to demonstrate the non-existence of physical objects looking a certain way. They were intended to lead to a rejection of the *reductio* supposition, in the context of retaining the remaining antededent suppositions. For example, considering Proposition 6 above, Euclid assumes that angle ABC=angle ACB, and for the *reductio*, he supposes side AB≠side AC. On deducing the contradiction, he rejects the latter supposition.

In contrast, for the IFs we have a proof of contradiction, from which it follows that no such 3-D object, one having the properties that the 3-D content has, exists. BUT at the same time the premisses are a "faithful" or "correct" report of the 3-D content, of the how-it-seems, its seemingness. Is this a kind of INTERNAL "truth"?

I don't have a problem with internal "truth", as long as it isn't conflated with the real thing, truth without the scare quotes. Sherlock Holmes lived in Baker Street, except not truly, because no-one of the name lived there. The internal truth here derives from words to that effect being included in a declared piece of fiction, or following from same, or satisfying some other constraints, declared or tacit. Humans have a great liking for playing with content without commitment to its truth: novels, films, paintings, diagrams, Impossible Figures (even mathematics, I would venture to claim). To avoid commitment to truth all you have to do is to qualify the narrative with "Once upon a time …". For this reason I weary of having to explain to people of a phenomenological persuasion: "But it isn't really TRUE!!". And I urge the attempt to avoid describing internal truth as a kind of truth, or you will get locked into your own perspective. The temptation is there, I acknowledge, it derives from needing an internal standard separating as Holmes' residence Baker Street from, say, Trafalgar Square (otherwise, chaos). But the internal standards are generally weaker than (external) truth, and subject to a greater level of convention.

So we have two different sorts of proof of contradiction, I suggest, but with overlaps. Euclid's *reductio* aims to prove a Proposition by rejecting its opposite as leading to a contradiction. It does not seek to demonstrate a paradox. The Steps proof does aim at showing that a certain thing does not exist: but in a sense this is secondary to the demonstration of the contradiction as a coherent contradictory narrative. Ipso facto the premisses of the Steps proof can be read off, since the premisses report how it **seems**. In *this* sense its premisses are co-exact, I would say: the co-exactness of premisses as the guarantee of their correctness is to be read off into the narrative. Thus, the attitude to contradiction is different, the Steps is a demonstration that humans willingly entertain contradictory contents.

But there is another difference too. It isn't obvious that there is the scope for *variation* in premisses of the Steps, *i.e.* that there is in a *reductio* in Manders' sense. Certainly Diagrams can be made of different sizes and orientations, but it isn't clear that there is the sort of scope for variation that Proposition 16 shows.

One final similarity is the 3-D aspect. The Euclidean examples above do not involve projection into 3-D, though Euclid did not shirk 3-D: from Book XI on there is discussed solid geometry. One difference here, however, is that a large class of *occlusion paradoxes*, namely paradoxes that depend on occlusion for their paradoxicality, are not addressed by Euclid. Needless to say, however, occlusion is an important mechanism of 3-D content.

9. Conclusion

We have discussed three contrasting ways in which words and geometrical diagrams relate to one another. In particular, we have seen that Manders' analysis provides us with tools for analysing Euclid's use of proof by contradiction. We also see that this manifests itself in a difference with proof by contradiction in the case of Impossible Figures.

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